

Intersection of G-structures of first or second order¹

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Abstract. The concept of G-structure of first or second order includes the majority of geometrical structures that can be defined on a manifold. We analyse the intersection of G-structures with different kinds of G. In particular, we discuss the relations among conformal, (semi) Riemannian, volume and projective geometrical structures on a fixed manifold. Some consequences are pointed out relative to geometrical aspects of General Relativity.

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1. Introduction and motivation

I will begin by explaining how I became interested in this topic. When I began my doctoral thesis I read an article by J. Ehlers ([3]) in which he developed an axiomatic approach to General Relativity theory. This approach is carried out in a paper by Ehlers et al. ([2]).

When we read these papers we realize that the geometry of the space-time has a conformal Lorentzian structure that explains the phenomenon of light propagation. In order to obtain the Lorentzian metric structure of the space-time, just as it maintains the General Relativity theory, these authors carry out the following steps.

Firstly, they introduce a projective structure that represents the movement of test free particles, suffering gravitational effects only. Then, a compatibility condition is necessary to link the projective and conformal structures in order to obtain a symmetric linear connection, the so called Weyl connection. This mathematical condition is physically justified by the experimental fact that a particle with mass can be made to chase a photon arbitrarily closely.

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Finally, they introduce the axiom which affirms that the Ricci tensor of the connection must be symmetric so that the Weyl connection becomes the linear connection associated with a Lorentzian metric. Once this axiom at hand has been physically justified, they induce the existence of a Lorentzian metric structure, uniquely determined except for a constant factor.

The introduction of the projective structure as an axiom did not totally convince me. On the one hand, when we have a conformal structure we are very near to a metric structure. Essentially, it is only necessary to fix a function on the manifold, the so called conformal factor. On the other hand, the physical concept of the test free particle was not very clear for me at that time.

The results, that I will show here, prove that given a volume structure and a conformal structure, we obtain a uniquely determined metric structure. This suggests that we must investigate the physical motivation in order to introduce a volume structure as a component of the space-time geometry.

2. A theorem about reductions of a principal bundle

We understand a manifold M as a C^∞ , second countable, manifold of dimension n . We give a theorem about the intersection of reductions of a principal bundle. Let G be a Lie group and let $P(M, G)$ be a principal bundle over M with G as structure group. If we have a subgroup H of G , we define a *reduction* $Q(M, H)$ of P , or an *H-reduction*, as a subbundle Q of P with H as structure group.

Theorem 2.1. *Let H, K be two closed subgroups of G such that $G = HK$, i.e., $\forall a \in G, \exists b \in H, c \in K: a = bc$. Let $Q(M, H)$ and $R(M, K)$ be two reductions of $P(M, G)$.*

Then, $Q \cap R$ is a reduced bundle of P , with $H \cap K$ as structure group.

Proof. There exists a bijective correspondence between the K -reductions of a principal bundle, $P(M, G)$, and the functions f of P into G/K verifying that $f \circ \mathcal{R}_a^P = \mu_{a^{-1}} \circ f, \forall a \in G$, with \mathcal{R}^P being the principal right action of G on P and with $\mu: G \times (G/K) \rightarrow G/K, \mu(a, bK) \equiv \mu_a(bK) := abK$. Thus $R \equiv f^{-1}(\{K\})$ becomes the K -reduction corresponding to f . We can prove, under the conditions of the theorem, that the application $\rho: G/K \rightarrow H/(H \cap K), \rho(aK) := b(H \cap K)$, with $b^{-1}a \in K$, is a bijection and ρ^{-1} is the application that maps $b(H \cap K)$ into bK . It follows of ([4, Chapter II, Proposition 4.4 (a)]) that ρ^{-1} is an immersion and then ρ is a diffeomorphism. We obtain the function $\rho \circ f|_Q: Q \rightarrow H/(H \cap K)$ which verifies

$$\rho \circ f|_Q \circ \mathcal{R}_b^Q = \tilde{\mu}_{b^{-1}} \circ \rho \circ f|_Q, \forall b \in H,$$

with \mathcal{R}^Q being the principal right action of H on Q and with $\tilde{\mu}: H \times (H/(H \cap K)) \rightarrow H/(H \cap K), \tilde{\mu}(b, c(H \cap K)) \equiv \tilde{\mu}_b(c(H \cap K)) := bc(H \cap K)$. Thus $(\rho \circ f|_Q)^{-1}(\{H \cap K\}) = Q \cap R$ is a reduction of Q , and the theorem follows on from the fact that a reduction, $Q \cap R$, of a reduction Q of P is a reduction of P . \square

Since a reduction of P with some subgroup of G as structure group extends uniquely to another reduction of P for any subgroup containing the former, we obtain the following corollary.

Corollary 2.2. *Let H, K be two closed subgroups of G such that $G = HK$. The $(H \cap K)$ -reductions of $P(M, G)$ are the intersections of H -reductions with K -reductions of P .*

3. Application to geometrical structures of first order

We can apply the previous results to the case when P is the linear frame bundle $LM(M, \text{GL}(n, \mathbb{R}))$ of M , i.e., the bundle of basis of the tangent space at every point of M or, equivalently, the bundle of 1-jets at $0 \in \mathbb{R}^n$ of inverse of charts of M . A G -structure of first order is a G -reduction of LM , with G being a subgroup of $\text{GL}(n, \mathbb{R})$. We are interested in the following types of G -structures.

Let η be the standard scalar product on \mathbb{R}^n of a fixed signature. The $O^\eta(n)$ -structures or (*semi*) *Riemannian* structures are the reductions of LM with subgroup

$$O^\eta(n) := \{a \in \text{GL}(n, \mathbb{R}) : a^\dagger a = I_n\},$$

where I_n is the identity matrix in $\text{GL}(n, \mathbb{R})$ and a^\dagger is the unique matrix such that $\eta(v, a^\dagger w) = \eta(av, w)$, $\forall v, w \in \mathbb{R}^n$. It is clear that an $O^\eta(n)$ -structure on M determines and is determined by a tensor metric g on M , this one whose orthonormal basis on $T_m M$ at every $m \in M$ are the elements of LM of the fiber over m which belong to the $O^\eta(n)$ -structure.

The $\text{CO}^\eta(n)$ -structures or *conformal* structures are the reductions of LM with subgroup

$$\text{CO}^\eta(n) := \{a \in \text{GL}(n, \mathbb{R}) : a^\dagger a = kI_n, k > 0\}.$$

A $\text{CO}^\eta(n)$ -structure on M determines and is determined by a set $[g]$ of metric tensors which are proportional by a positive factor to a given metric tensor g on M (i.e., $g' \in [g]$ if and only if $g' = \omega g$, with $\omega: M \rightarrow \mathbb{R}^+$). The orthonormal basis on $T_m M$ for every $g' \in [g]$ are the elements of LM of the fiber over m , which belong to the $\text{CO}^\eta(n)$ -structure.

The $\text{SL}^\pm(n)$ -structures or *volume* structures are the reductions of LM with subgroup

$$\text{SL}^\pm(n) := \{a \in \text{GL}(n, \mathbb{R}) : |\det(a)| = 1\}.$$

An $\text{SL}^\pm(n)$ -structure on M can be understood as a selection, for every point $m \in M$, of a maximal set of basis of $T_m M$ with the same unoriented volume, in the sense of linear algebra. It determines a family of local volume forms, which when applied to a basis belonging to the $\text{SL}^\pm(n)$ -structure gives $+1$ or -1 . The existence of volume structures does not depend on the orientability of M as in the case of $\text{SL}(n)$ -structures, but if M is an orientable manifold a $\text{SL}^\pm(n)$ -structure on M determines and is determined by a global volume form, unique except for the sign.

Since $\text{CO}(n) \cap \text{SL}^\pm(n) = \text{O}(n)$ and $\text{GL}(n, \mathbb{R}) = \text{CO}(n)\text{SL}^\pm(n)$ because $a = (|\det(a)|^{1/n}I) (|\det(a)|^{-1/n}a)$, $\forall a \in \text{GL}(n, \mathbb{R})$, we obtain the following result as an application of the Corollary 2.2 (see the comments in the final part of the Section 1).

Theorem 3.1. *The (semi) Riemannian structures on M are the intersections of conformal and volume structures on M .*

4. Application to geometrical structures of second order

Now, we apply the results of Section 2 to the case when P is the *second order frame bundle of M* , $F^2M(M, \text{G}^2(n))$, i.e., the bundle of 2-jets at $0 \in \mathbb{R}^n$ of inverse of charts of M . The group $\text{G}^2(n)$ is the group of 2-jets at $0 \in \mathbb{R}^n$ of (local) diffeomorphisms ϕ of \mathbb{R}^n , with $\phi(0) = 0$. It is described as the semidirect product of Lie groups $\text{GL}(n, \mathbb{R}) \rtimes \text{S}^2(n)$, with the product law

$$(a, t) \cdot (a', t') := (aa', a'^{-1}t(a', a') + t'),$$

by the isomorphism that maps $j_0^2(\phi)$ into $(D\phi|_0, D\phi|_0^{-1}D^2\phi|_0)$, where we denote $\text{S}^2(n)$ for the set of symmetric bilinear maps of $\mathbb{R}^n \times \mathbb{R}^n$ into \mathbb{R}^n . We can identify F^2M with the first prolongation $(LM)_1$ of LM (see [1]).

A G -structure of second order on M is a G -reduction of $F^2(M)$, with G being a subgroup of $\text{G}^2(n) \equiv \text{GL}(n, \mathbb{R}) \rtimes \text{S}^2(n)$. The symmetric linear connections of M are examples of G -structures of second order (see [5, 6]), when $G = \text{GL}(n, \mathbb{R}) \rtimes \{0\}$. The projective structures on M are also examples of it (see [5, 6]), when $G = \text{GL}(n, \mathbb{R}) \rtimes \mathfrak{p}$, where

$$\mathfrak{p} := \{t \in \text{S}^2(n) : t_{jk}^i = \delta_j^i p_k + \delta_k^i p_j, \text{ for some } (p_i) \in \mathbb{R}^{n*}\}.$$

More examples of G -structures of second order (see [7]) are the first prolongation of the H -structures of first order which admit a symmetric linear connection, just as the (semi) Riemannian, conformal and volume structures. In these cases $G = H \rtimes \mathfrak{h}_1$, where \mathfrak{h}_1 is the first prolongation of the Lie algebra \mathfrak{h} of H .

Now, we can apply the Theorem 2.1 to the following cases.

Theorem 4.1. *The intersection of a projective structure on M and the first prolongation of a volume structure on M gives a symmetric linear connection of M .*

Proof. If $t \in \mathfrak{p} \cap \mathfrak{sl}(n)_1$, since $\mathfrak{sl}(n)_1 = \{t \in \text{S}^2(n) : t_{hk}^h = 0\}$, it follows that $0 = t_{hk}^h = \delta_h^h p_k + \delta_k^h p_h = (n+1)p_k$, $\forall k$, thus $t = 0$. This shows that

$$(\text{GL}(n, \mathbb{R}) \rtimes \mathfrak{p}) \cap (\text{SL}^\pm(n) \rtimes \mathfrak{sl}(n)_1) = \text{SL}^\pm(n) \rtimes \{0\}.$$

Now, we will show that

$$\text{GL}(n, \mathbb{R}) \rtimes \text{S}^2(n) = (\text{GL}(n, \mathbb{R}) \rtimes \mathfrak{p}) \cdot (\text{SL}^\pm(n) \rtimes \mathfrak{sl}(n)_1).$$

It follows from $(a, t) = (a, r) \cdot (I, s)$, $\forall (a, t) \in \text{GL}(n, \mathbb{R}) \rtimes \text{S}^2(n)$, with

$$r_{jk}^i := \frac{1}{n+1}(\delta_j^i t_{hk}^h + \delta_k^i t_{hj}^h), \quad s_{jk}^i := t_{jk}^i - \frac{1}{n+1}(\delta_j^i t_{hk}^h + \delta_k^i t_{hj}^h).$$

Then, by Theorem 2.1, we obtain that the intersection of a projective structure and the first prolongation of a volume structure is an $(\text{SL}^\pm(n) \rtimes \{0\})$ -structure of second order. This extends naturally to a $(\text{GL}(n, \mathbb{R}) \rtimes \{0\})$ -structure of second order. \square

Theorem 4.2. *The intersection of the first prolongation of a conformal structure on M and the first prolongation of a volume structure on M gives a symmetric linear connection of M .*

Proof. The first prolongation of the Lie algebra $\mathfrak{co}^\eta(n)$ of $\text{CO}^\eta(n)$ is

$$\mathfrak{co}^\eta(n)_1 = \{t \in \text{S}^2(n) : t_{jk}^i = \delta_j^i q_k + \delta_k^i q_j - \eta^{is} \eta_{jk} q_s, \text{ with } (q_i) \in \mathbb{R}^{n*}\}.$$

From $t \in \mathfrak{co}^\eta(n)_1 \cap \mathfrak{sl}(n)_1$, it follows $0 = t_{hk}^h = \delta_h^h q_k + \delta_k^h q_h - \eta^{hs} \eta_{hk} q_s = nq_k$, $\forall k$, thus $t = 0$. This shows that

$$(\text{CO}^\eta(n) \rtimes \mathfrak{co}^\eta(n)_1) \cap (\text{SL}^\pm(n) \rtimes \mathfrak{sl}(n)_1) = \text{O}^\eta(n) \rtimes \{0\}.$$

Now, we will show that

$$\text{GL}(n, \mathbb{R}) \rtimes \text{S}^2(n) = (\text{CO}^\eta(n) \rtimes \mathfrak{co}^\eta(n)_1) \cdot (\text{SL}^\pm(n) \rtimes \mathfrak{sl}(n)_1).$$

It follows from $(a, t) = (a, r) \cdot (I, s)$, $\forall (a, t) \in \text{GL}(n, \mathbb{R}) \rtimes \text{S}^2(n)$, with

$$r_{jk}^i := \frac{1}{n}(\delta_j^i t_{hk}^h + \delta_k^i t_{hj}^h - \eta^{il} \eta_{jk} t_{hl}^h),$$

$$s_{jk}^i := t_{jk}^i - \frac{1}{n}(\delta_j^i t_{hk}^h + \delta_k^i t_{hj}^h - \eta^{il} \eta_{jk} t_{hl}^h).$$

By Theorem 2.1, we obtain that the intersection of the first prolongation of a conformal structure and the first prolongation of a volume structure is an $(\text{O}^\eta(n) \rtimes \{0\})$ -structure of second order. This extends naturally to a $(\text{GL}(n, \mathbb{R}) \rtimes \{0\})$ -structure of second order. \square

We can not apply Theorem 2.1 in the case of the intersection of a projective structure and the first prolongation of a conformal structure because with their structure groups, $\text{GL}(n, \mathbb{R}) \rtimes \mathfrak{p}$ and $\text{CO}^\eta(n) \rtimes \mathfrak{co}^\eta(n)_1$, we can not generate by products the total group $\text{GL}(n, \mathbb{R}) \rtimes \text{S}^2(n)$. We need some compatibility condition to assure that the intersection is a well defined reduction (see the comments in Section 1).

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